From conformal invariance towards dynamical symmetries of the collisionless Boltzmann equation

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Abstract

Dynamical symmetries of the collisionless Boltzmann transport equation, or Vlasov equation, but under the influence of an external driving force, are derived from non-standard representations of the $2D$ conformal algebra. In the case without external forces, the symmetry of the conformally invariant transport equation is first generalised by considering the particle momentum as an independent variables. This new conformal representation can be further extended to include an external force. The construction and possible physical applications are outlined.

1 Introduction

The *Boltzmann transport equation* (BTE) [1, 14, 7, 8] furnishes a semi-classical description of the effects of particle transport, including the influence of external forces, on the effective single-particle distribution function $f = f(t, \boldsymbol{r}, \boldsymbol{p})$ of a small cell in phase phase, centred at position $\boldsymbol{r}$ and momentum $\boldsymbol{p}$. For a system with identical particles of mass m, the Boltzmann equation reads

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{F} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = \left(\frac{\partial f}{\partial t} \right)_{\rm coll} . \tag{1.1}$$

Here, $\mathrm{d}N = f(t, \boldsymbol{r}, \boldsymbol{p},)\mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{p}$ is the number of particles in a cell of phase volume $\mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{p}$, centred at position $\boldsymbol{r}$ and momentum $\boldsymbol{p}$ [8]. In addition, $\boldsymbol{F} = \boldsymbol{F}(t, \boldsymbol{r})$ is the force field acting on the particles in the fluid. The term on the right-hand-side is added to describe the effect of collisions between particles. It is a statistical term and requires knowledge of the statistics the particles obey, like the Maxwell-Boltzmann, Fermi-Dirac or Bose-Einstein distributions. In his famous 'Stoßzahlansatz' (or hypothesis of molecular chaos), Boltzmann obtained an explicit form for it. In a modern notation, for example for an interacting Fermi gas, where a particle from a state with momentum $\boldsymbol{p}$ is scattered to a state with momentum $\boldsymbol{p}'$, whereas a second particle is scattered from a momentum $\boldsymbol{q}$ to a momentum $\boldsymbol{q}'$, the collision term reads

$$\begin{aligned} \left(\frac{\partial f}{\partial t} \right)_{\rm coll} &= - \int \mathrm{d}\boldsymbol{p}'\mathrm{d}\boldsymbol{q}\mathrm{d}\boldsymbol{q}'\; w(\{\boldsymbol{p}, \boldsymbol{q}\} \to \{\boldsymbol{p}', \boldsymbol{q}'\}) \\ &\quad \times [f(\boldsymbol{p})f(\boldsymbol{q})(1 - f(\boldsymbol{p}'))(1 - f(\boldsymbol{q}')) - f(\boldsymbol{p}')f(\boldsymbol{q}')(1 - f(\boldsymbol{p}))(1 - f(\boldsymbol{q}))] \end{aligned}$$

where $w(\{\boldsymbol{pq}\} \to \{\boldsymbol{p}'\boldsymbol{q}'\})$ is the normalised transition probability from the two-particle state with momenta $\{\boldsymbol{p}, \boldsymbol{q}\}$ to the state labelled by $\{\boldsymbol{p}', \boldsymbol{q}'\}$. Clearly, solving this widely studied equation is a very difficult task. It might be hoped that symmetries could be helpful. The equation without the collision term is known as the *Vlasov equation* [17]. Relationship with Landau damping and a physicists' derivation can be found in [18, 5]. In this work, we shall explore a class of symmetries of the (collisionless) BTE.

Throughout, we shall restrict to $d = 1$ space dimension[1]. We start from a non-standard representation, isomorphic to the infinite-dimensional Lie algebra of conformal transformations in $d = 2$ dimensions.[2] This Lie algebra is spanned by the generators $\langle X_n, Y_n \rangle_{n \in \mathbb{Z}}$ and can be defined from the commutators [9, 12]

$$[X_n, X_m] = (n - m)X_{n+m}, \quad [X_n, Y_m] = (n - m)Y_{n+m}, \quad [Y_n, Y_m] = \mu(n - m)Y_{n+m} \tag{1.2}$$

where μ is a parameter. An explicit realisation in terms of time-space transformation is [9, 12]:

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - \mu^{-1}[(t + \mu r)^{n+1} - t^{n+1}]\partial_r - (n + 1)xt^n - (n + 1)\frac{\gamma}{\mu}[(t + \mu r)^n - t^n] \\ Y_n &= -(t + \mu r)^{n+1}\partial_r - (n + 1)\gamma(t + \mu r)^n \end{aligned} \tag{1.3}$$

[1] By analogy with other constructions of local scale symmetries, see [9, 4, 15, 13] and especially [12] and refs. therein, we expect a straightforward extension of the results reported here to $d > 1$. Since we shall construct here a finite-dimensional Lie algebra of dynamical conformal symmetries of the $1D$ collisionless BTE, one should indeed expect that an extension to $d > 1$ exists. That symmetry algebra should contain three generators $X_{\pm 1,0}$, along with a vector of generators $\boldsymbol{Y}_n$ and also spatial rotations.

[2] For the sake of clarify, we shall adopt the following convention of terminology: the infinite-dimensional Lie algebra $\langle X_n, Y_n \rangle_{n \in \mathbb{Z}}$ will be called a (centreless) '*conformal Virasoro algebra*'. Its maximal finite-dimensional sub-algebra $\langle X_n, Y_n \rangle_{n \in \{-1,0,1\}}$ will be called a '*conformal algebra*'.

such that μ^{-1} can be interpreted as a velocity ('speed of light/sound') and where x, γ are constants.[3] Writing $X_n = \ell_n + \bar{\ell}_n$ and $Y_n = \mu^{-1}\bar{\ell}_n$, where the generators $\langle \ell_n, \bar{\ell}_n \rangle_{n\in\mathbb{Z}}$ satisfy $[\ell_n, \ell_m] = (n-m)\ell_{n+m}, [\bar{\ell}_n, \bar{\ell}_m] = (n-m)\bar{\ell}_{n+m}, [\ell_n, \bar{\ell}_m] = 0$, it can be seen that, provided $\mu \neq 0$, the above Lie algebra (1.2) is isomorphic to a pair of Virasoro algebras $\mathfrak{vect}(S^1) \oplus \mathfrak{vect}(S^1)$ with a vanishing central charge. However, this isomorphism does not imply that physical systems described by two different representations of the conformal Virasoro algebra, or the conformal algebra, with commutators (1.2), were trivially related. For example, it is well-known that if one uses the generators of the standard representation of conformal invariance or else the non-standard representation (1.4) in order to find co-variant two-point functions, the resulting scaling forms are different [9].

Now, consider the maximal finite-dimensional sub-algebra $\langle X_{\pm 1,0}, Y_{\pm 1,0} \rangle$, which for $\mu \neq 0$ in turn is isomorphic to the direct sum $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. The explicit realisation follows from from (1.3)

$$\begin{aligned} X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - r\partial_r - x, \quad\quad X_1 = -t^2\partial_t - 2tr\partial_r - \mu r^2\partial_r - 2xt - 2\gamma r \\ Y_{-1} &= -\partial_r, \quad Y_0 = -t\partial_r - \mu r\partial_r - \gamma, \quad Y_1 = -t^2\partial_r - 2\mu tr\partial_r - \mu^2 r^2\partial_r - 2\gamma t - 2\gamma\mu r \end{aligned} \tag{1.4}$$

Here, the generators X_{-1}, Y_{-1} describe time- and space-translations, Y_0 is a (conformal) Galilei transformation,[4] X_0 gives the dynamical scaling $t \mapsto \lambda t$ of $r \mapsto \lambda r$ (with $\lambda \in \mathbb{R}$) such that the so-called 'dynamical exponent' $z = 1$ since both time and space are re-scaled in the same way and finally X_{+1}, Y_{+1} give 'special' conformal transformations. In the context of statistical mechanics of conformally invariant phase transitions, one characterises co-variant quasi-primary scaling operators through the invariant parameters (x, μ, γ), where x is the scaling dimension.

Finally, the finite-dimensional representation (1.4) acts as a dynamical symmetry on the equation of motion

$$\hat{S}\phi(t,r) = (-\mu\partial_t + \partial_r)\phi(t,r) = 0. \tag{1.5}$$

in the sense that a solution ϕ of $\hat{S}\phi = 0$ is mapped onto another solution of the same equation. Indeed, it is easily checked that $[\hat{S}, Y_{\pm 1,0}] = [\hat{S}, X_{-1}] = 0$ and

$$[\hat{S}, X_0] = -\hat{S}, \quad [\hat{S}, X_1] = -2t\hat{S} + 2(\mu x - \gamma) \tag{1.6}$$

It follows that for fields ϕ with scaling dimensions $x_\phi = x = \gamma/\mu$ the algebra (1.4) really leaves the solution space of the equation (1.5) invariant.[5]

In order to return to the Boltzmann equation, we consider eq. (1.5) in the form

$$\hat{L}f = (\mu\partial_t + v\partial_r)f(t,r,v) = 0 \tag{1.7}$$

where $f = f(t,r,v)$ is interpreted as a single-particle distribution function and where we consider v as an additional variable. Eq. (1.7) is a simple Boltzmann (or Vlasov) equation, without

[3] The contraction $\mu \to 0$ of (1.3) produces the non-semi-simple 'altern-Virasoro algebra' $\mathfrak{altv}(1)$ (but without central charges). Its maximal finite-dimensional sub-algebra is the conformal galilean algebra $\mathfrak{alt}(1) \equiv \mathrm{CGA}(1)$ [9, 10], see also [4, 15]. The CGA(d) is non-isomorphic to either the standard Galilei algebra or else the Schrödinger algebra.

[4] Since the commutator $[Y_0, Y_{-1}]$ does not vanish and does not give a central element of the Lie algebra (1.2), its structure is fundamentally different from algebras containing the usual Galilei algebra as a sub-algebra.

[5] Since $[\hat{S}, X_n] = -(n+1)t^n\hat{S} + n(n+1)t^{n-1}(\mu x - \gamma)$ and $[\hat{S}, Y_n] = 0$, for all $n \in \mathbb{Z}$, this symmetry extends to the centreless Virasoro algebra (1.2).

an external force and without a collision term, and in one space dimension. From (1.6), with v fixed (and normalised to $v = 1$), its solution space is conformally invariant[6]. In section 2, we shall generalise the above representation of the conformal algebra to the situation with v as a further variable. In section 3, we shall further extend this to the case when an external force $F = F(t, r, v)$, possibly depending on time, spatial position and velocity, is included. The aim of these calculations is to determine which situations of potential physical interest with a non-trivial conformal symmetry might be identified. This explorative study aims at identifying lines for further study, which might lead later to a more comprehensive understanding of the possible symmetries of Boltzmann equations. Taking into account the collision term is left for future work. We shall concentrate on $d = 1$ space dimension throughout. Conclusions and final comments are given in section 4.

2 Collisionless Boltzmann equation without external forces

In our construction of conformal dynamical symmetries of the $1D$ collisionless BTE, we shall often meet Lie algebras of a certain structure. These will be isomorphic to the two-dimensional conformal algebra.

Proposition 1: *The Lie algebra $\langle X_n, Y_n\rangle_{n\in\mathbb{Z}}$ defined by the commutators*

$$[X_n, X_m] = (n-m)X_{n+m}, \quad [X_n, Y_m] = (n-m)Y_{n+m}, \quad [Y_n, Y_m] = (n-m)\left(kX_{n+m} + qY_{n+m}\right) \tag{2.1}$$

where k, q are constants, is isomorphic to the pair of centreless Viraso algebras $\mathfrak{vect}(S^1) \oplus \mathfrak{vect}(S^1)$.

Proof: For either $k = 0$ or $q = 0$ this is either evident or else has already been seen in section 1. In the other case, consider the change of basis $X_n = \ell_n + \bar{\ell}_n$ and $Y_n = \alpha\ell_n - \beta\bar{\ell}_n$ where $\ell_n, \bar{\ell}_n$ are two families of commuting generators of $\mathfrak{vect}(S^1)$ and α and β are constants such that $\alpha + \beta \neq 0$. It then follows $k = \alpha\beta$ and $q = \alpha - \beta$. q.e.d.

This implies in particular the isomorphism of the maximal finite-dimensional sub-algebras, or 'conformal algebras' in the terminology chosen here. By definition, this 'conformal algebra' obeys the commutators (2.1), but with $n, m \in \{-1, 0, 1\}$.

Our construction of dynamical symmetries of the equation (1.7) follows the lines of the construction of local scale-invariance in time-dependent critical phenomena [9]. The physically motivated requirements are: First of all it is clear that the equation is invariant under time-translations:

$$X_{-1} = -\partial_t, \quad [\hat{L}, X_{-1}] = 0 \tag{2.2}$$

Some kind of dynamical scaling must be present as well. Its most general form is

$$X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - x, \quad [\hat{L}, X_0] = -\hat{L}. \tag{2.3}$$

Whenever, the dynamical exponent $z \neq 1$, we shall find an explicit dependence on v. In general,

[6]With respect to eq. (1.5), $\mu \mapsto -\mu$ was replaced. This change must also be made in the generators (1.4) and commutators (1.2).

we look for a family of generators X_n, for which we make the ansatz

$$X_n = -a_n(t,r,v)\partial_t - b_n(t,r,v)\partial_r - c_n(t,r,v)\partial_v - d_n(t,r,v). \tag{2.4}$$

We shall find X_n from the following three conditions (throughout, we use the notations $\partial_t f = \dot{f}, \partial_r f = f'$):

1. X_n must be a symmetry for the equation (1.7), hence $[\hat{L}, X_n] = \lambda_n \hat{L}$. This gives

$$\begin{aligned} &\mu\dot{a}_n + v a_n' + \mu\lambda_n = 0, \quad \mu\dot{b}_n + v b_n' - c_n + \lambda_n v = 0 \\ &\mu\dot{c}_n + v c_n' = 0, \quad \mu\dot{d}_n + v d_n' = 0. \end{aligned} \tag{2.5}$$

2. The generator X_0 is assumed to be in the Cartan sub-algebra, hence $[X_n, X_0] = \alpha_{n,0} X_n$. It follows

$$(1+\alpha_{n,0})a_n - t\dot{a}_1 - \frac{r}{z}a_n' - \frac{1-z}{z}v\partial_v a_n = 0 \tag{2.6}$$

$$(1/z+\alpha_{n,0})b_n - t\dot{b}_n - \frac{r}{z}b_n' - \frac{1-z}{z}v\partial_v b_n = 0 \tag{2.7}$$

$$((1-z)/z+\alpha_{n,0})c_n - t\dot{c}_1 - \frac{r}{z}c_n' - \frac{1-z}{z}v\partial_v c_n = 0 \tag{2.8}$$

$$\alpha_{n,0}d_n - t\dot{d}_n - \frac{r}{z}d_n' - \frac{1-z}{z}v\partial_v d_n = 0. \tag{2.9}$$

3. The action of X_{-1} is as a lowering operator, hence $[X_n, X_{-1}] = \alpha_{n,-1}X_{n-1}$. It follows

$$\begin{aligned} \dot{a}_n &= \alpha_{n,-1}t, \quad \dot{b}_n = \alpha_{n,-1}r/z \\ \dot{c}_n &= \alpha_{n,-1}v(1-z)/z, \quad \dot{d}_n = \alpha_{n,-1}x/z. \end{aligned} \tag{2.10}$$

These conditions, combined with the following initial conditions:

$$\begin{aligned} a_0 &= t, \quad b_0 = \frac{r}{z}, \quad c_0 = \frac{1-z}{z}v, \quad d_0 = x \\ a_{-1} &= 1, \quad b_{-1} = 0, \quad c_{-1} = 0, \quad d_{-1} = 0. \end{aligned} \tag{2.11}$$

must be sufficient for determination of all admissible forms of X_n.

In the special case $n = 1$, we have $\alpha_{1,0} = 1$ and find the most general form of X_1 as a symmetry of (1.7) as follows:[7]

$$X_1 = -a_1(t,r,v)\partial_t - b_1(t,r,v)\partial_r - c_1(t,r,v)\partial_v - d_1(t,r,v) \tag{2.12}$$

and

$$a_1(t,r,v) = t^2 + A_{12}r^2v^{-2} + A_{110}rv^{\frac{2z-1}{1-z}} + A_{100}v^{\frac{2z}{1-z}} \tag{2.13}$$

$$b_1(t,r,v) = \frac{2}{z}tr + \left(\frac{A_{12}}{\mu} + \frac{z-2}{z}\mu\right)r^2v^{-1} + B_{110}rv^{\frac{z}{1-z}} + B_{100}v^{\frac{z+1}{1-z}} \tag{2.14}$$

$$c_1(t,r,v) = \frac{2}{z}(1-z)(vt-\mu r) + (B_{110} - \frac{A_{110}}{\mu})v^{\frac{z}{1-z}} \tag{2.15}$$

$$d_1(t,r,v) = \frac{2}{z}xt - \frac{2}{z}\mu xrv^{-1} + D_0v^{\frac{z}{1-z}} \tag{2.16}$$

[7] The requirement that $\langle X_{\pm 1,0}\rangle$ close into the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ fixes $\alpha_{1,-1} = 2$.

with a certain set of undetermined constants.

For conformal invariance, a family of generators Y_n must also be found. Its construction is straightforward if the explicit form of Y_{-1} is known. Really X_1 must act as a raising operator, in both hierarchies, such that [9]

$$[X_1, Y_{-1}] \sim Y_0, \quad [X_1, Y_0] \sim Y_1. \tag{2.17}$$

which implies that $[Y_{-1}, [Y_{-1}, X_1]] \sim Y_{-1}$. However, the usual realization of $Y_{-1} = -\partial_r$ as space translations does *not* work, since if we set all undetermined constants in eq. (2.12) to zero, one would have $[Y_{-1}, [Y_{-1}, X_1]] \sim v^{-1} Y_{-1}$. It is better to work with the form

$$Y_{-1} = -v\partial_r. \tag{2.18}$$

as we shall do from now on.

We first consider the special case, when all the constants in the expression (2.12) for X_1 vanish:

Case A: $A_{12} = A_{110} = A_{100} = B_{110} = B_{100} = D_0 = 0$.

Proposition 2: *The six generators*

$$\begin{aligned}
X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - \frac{x}{z} \\
X_1 &= -t^2\partial_t - \left(\frac{2}{z}tr + \frac{z-2}{z}\mu r^2 v^{-1}\right)\partial_r - \frac{2(1-z)}{z}(vt - \mu r)\partial_v - \frac{2}{z}xt + \frac{2}{z}\mu x r v^{-1} \\
Y_{-1} &= -v\partial_r, \quad Y_0 = -(tv - \frac{\mu}{z}r)\partial_r - \frac{z-1}{z}\mu v\partial_v + \mu\frac{x}{z} \\
Y_1 &= -\left(t^2 v - \frac{2}{z}\mu t r - \frac{z-2}{z}\mu^2 r^2 v^{-1}\right)\partial_r - \frac{2}{z}(z-1)\mu(vt - \mu r)\partial_v \\
&\quad + \frac{2}{z}\mu x t - \frac{2}{z}\mu^2 x r v^{-1}
\end{aligned} \tag{2.19}$$

span a representation of the conformal algebra (1.2), which acts as dynamical symmetry algebra of the equation (1.7), for arbitrary dynamical exponent z.

Proof: It is readily checked that the generators (2.19) satisfy the commutation relations (1.2), with $\mu \mapsto -\mu$. On the other hand, for any $f = f(t, r, v)$, one has

$$\begin{aligned}
&[\hat{L}, X_{-1}] = [\hat{L}, Y_{-1}] = [\hat{L}, Y_0] = [\hat{L}, Y_1] = 0 \\
&[\hat{L}, X_0] = -\hat{L}, \quad [\hat{L}, X_1] = -2t\hat{L},
\end{aligned}$$

which establishes the asserted dynamical symmetry. q.e.d.

Next, we treat the general case, when all the constants are non-zero:

Case B: $A_{12} \neq 0, A_{110} \neq 0, A_{100} \neq 0, B_{110} \neq 0, B_{100} \neq 0, D_0 \neq 0$.

Then the generators are modified as follows:

$$
\begin{aligned}
\bar{X}_1 &= X_1 + \widetilde{X}_1 \\
\widetilde{X}_1 &= -\left(A_{12} r^2 v^{-2} + A_{110} r v^{\frac{2z-1}{1-z}} + A_{100} v^{\frac{2z}{1-z}}\right)\partial_t \\
&\quad -\left(\frac{A_{12}}{\mu} r^2 v^{-1} + B_{110} r v^{\frac{z}{1-z}} + B_{100} v^{\frac{z+1}{1-z}}\right)\partial_r \\
&\quad -(B_{110} - \frac{A_{110}}{\mu}) v^{\frac{z}{1-z}}\partial_v - D_0 v^{\frac{z}{1-z}},
\end{aligned} \tag{2.20}
$$

$$
\begin{aligned}
\bar{Y}_0 &= Y_0 + \widetilde{Y}_0 \\
\widetilde{Y}_0 &= \frac{1}{2}[\widetilde{X}_1, Y_{-1}] \\
&= -(A_{12} r v^{-1} + \frac{1}{2} A_{110} v^{-1+1/(1-z)})\partial_t - \frac{1}{2\mu}(2A_{12} r + A_{110} v^{1/(1-z)})\partial_r.
\end{aligned} \tag{2.21}
$$

Now, computing

$$
[\bar{Y}_0, Y_{-1}] = -\mu Y_{-1} + A_{12} X_{-1} + \frac{A_{12}}{\mu} Y_{-1} \tag{2.22}
$$

we conclude that the cases $A_{12} = 0$ and $A_{12} \neq 0$ must be treated separately.

case B1: $A_{12} = 0$. It follows that the constants in (2.12) are given by:

$$
B_{110} = A_{110}/\mu, \quad A_{100} = \frac{A_{110}^2}{4\mu^2}, \quad B_{100} = \frac{A_{100}}{\mu} = \frac{A_{110}^2}{4\mu^3}, \quad D_0 = 0. \tag{2.23}
$$

Proposition 3: *Let $z \neq 1$ and A_{110} be arbitrary constants. Then the six generators*

$$
\begin{aligned}
\bar{X}_{-1} &= -\partial_t, \quad \bar{X}_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z} v\partial_v - \frac{x}{z} \\
\bar{X}_1 &= -(t^2 + A_{110} r v^{(2z-1)/(1-z)} + \frac{A_{110}^2}{4\mu^2} v^{2z/(1-z)})\partial_t \\
&\quad -\left(\frac{2}{z} tr + \frac{z-2}{z}\mu r^2 v^{-1} + \frac{A_{110}}{\mu} r v^{z/(1-z)} + \frac{A_{110}^2}{4\mu^3} v^{(z+1)/(1-z)}\right)\partial_r \\
&\quad -\frac{2(1-z)}{z}(vt - \mu r)\partial_v - \frac{2}{z} xt + \frac{2}{z}\mu x r v^{-1} \\
\bar{Y}_{-1} &= -v\partial_r \\
\bar{Y}_0 &= -\frac{A_{110}}{2} v^{z/(1-z)}\partial_t - (tv - \frac{\mu}{z} r + \frac{A_{110}}{2\mu} v^{1/(1-z)})\partial_r - \frac{z-1}{z}\mu v\partial_v + \mu\frac{x}{z} \\
\bar{Y}_1 &= -A_{110}(t v^{z/(1-z)} - \mu r v^{(2z-1)/(1-z)})\partial_t \\
&\quad -\left(t^2 v - \frac{2}{z}\mu t r - \frac{z-2}{z}\mu^2 r^2 v^{-1} + \frac{A_{110}}{\mu}(t v^{1/(1-z)} - \mu r v^{z/(1-z)})\right)\partial_r \\
&\quad -\frac{2}{z}(z-1)\mu(vt - \mu r)\partial_v + \frac{2}{z}\mu x t - \frac{2}{z}\mu^2 x r v^{-1}
\end{aligned} \tag{2.24}
$$

span a representation of the conformal algebra.[8] *These generators give more symmetries of the equation (1.7).*

[8] The above result of **case A** is recovered upon setting $A_{110} = 0$.

Proof: From the above, the commutator (1.2) are readily verified, with $\mu \mapsto -\mu$. For the dynamical symmetries, one checks the commutators

$$
\begin{aligned}
[\hat{L}, X_{-1}] &= [\hat{L}, Y_{\pm 1,0}] \;=\; 0 \\
[\hat{L}, X_0] &= -\hat{L}, \quad [\hat{L}, X_1] = -(2t + \frac{A_{110}}{\mu} v^{z/(1-z)})\hat{L}.
\end{aligned}
$$

which proves the assertion. q.e.d.

In contrast to the previous **case A**, the representation acting only on (t, r) but keeps v is a constant parameter, can no longer be obtained by simply setting $z = 1$. Rather, one must set $A_{110} = 0$ first and only then the limit $z \to 1$ is well-defined.

case B2: $A_{12} \neq 0, A_{110} \neq 0, B_{110} \neq 0, A_{100} \neq 0, B_{100} \neq 0, D_0 \neq 0$.

It turns out that for $A_{12} \neq 0$, the algebra also can be closed, but only if $A_{12} = \mu$ and $A_{110} = 0$ (then all others constants also vanish).

Proposition 4: *Let z be an arbitrary constant. Then the generators $\langle \mathcal{X}_{\pm 1,0}, \mathcal{Y}_{\pm 1,0} \rangle$, where*

$$
\begin{aligned}
\mathcal{X}_{-1} &= -\partial_t, \quad \mathcal{X}_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z} v\partial_v - \frac{x}{z} \\
\mathcal{X}_{-1} &= X_{-1}, \quad \mathcal{X}_0 = X_0 \\
\mathcal{X}_1 &= -\left(t^2 + \mu r^2 v^{-2}\right)\partial_t - \left(\frac{2}{z} tr + \frac{z + \mu(z-2)}{z} r^2 v^{-1}\right)\partial_r \\
&\quad - \frac{2(1-z)}{z}(vt - \mu r)\partial_v - \frac{2}{z} xt + \frac{2}{z}\mu x r v^{-1} \\
\mathcal{Y}_{-1} &= -v\partial_r \\
\mathcal{Y}_0 &= -\mu r v^{-1}\partial_t - \left(tv - (\frac{\mu}{z} - 1) r\right)\partial_r - \frac{z-1}{z}\mu v\partial_v + \mu\frac{x}{z} \\
\mathcal{Y}_1 &= -\mu\left(2trv^{-1} + (1-\mu) r^2 v^{-2}\right)\partial_t - \left(t^2 v - \frac{2}{z}(z-\mu) tr + \frac{z(1-\mu) - (z-2)\mu^2}{z} r^2 v^{-1}\right)\partial_r \\
&\quad - \frac{2}{z}(z-1)\mu(vt - \mu r)\partial_v + \frac{2}{z}\mu x t - \frac{2}{z}\mu^2 x r v^{-1}
\end{aligned}
\tag{2.25}
$$

close into a Lie algebra, with the following non-zero commutation relations

$$
\begin{aligned}
&[\mathcal{X}_n, \mathcal{X}_{n'}] = (n - n')\mathcal{X}_{n+n'}, \quad [\mathcal{X}_n, \mathcal{Y}_m] = (n - m)\mathcal{Y}_{n+m} \\
&[\mathcal{Y}_m, \mathcal{Y}_{m'}] = (m - m')\left(\mu\mathcal{X}_{m+m'} + (1-\mu)\mathcal{Y}_{m+m'}\right),
\end{aligned}
\tag{2.26}
$$

with $n, n', m, m' \in \{-1, 0, 1\}$. The algebra is isomorphic to the usual conformal algebra (1.2) and further extends the dynamical symmetries of the equation (1.7).

Proof: The commutation relation are directly verified. The isomorphism with the conformal algebra follows from Proposition 1. The requirement to have an symmetry algebra of equation (1.7) implies a relation between the constants k, q (called α, β in Proposition 1) and μ, namely $q = (k - \mu^2)/\mu$. In this case at hand, we have $k = \mu$, $q = 1 - \mu$. It is then verified that

$[\hat{L}, \mathcal{X}_{-1}] = [\hat{L}, \mathcal{Y}_{-1}] = 0$ and

$$
\begin{aligned}
[\hat{L}, \mathcal{X}_0] &= -\hat{L} \\
[\hat{L}, \mathcal{X}_1] &= -2(t + \frac{r}{z} v^{-1})\hat{L} \\
[\hat{L}, \mathcal{Y}_0] &= -(k/\mu)\hat{L} = -\hat{L} \\
[\hat{L}, \mathcal{Y}_1] &= -2\left(\frac{k}{\mu} t + \frac{k}{z\mu^2} r v^{-1}\right)\hat{L} = -2\left(t + \frac{1}{z\mu} r v^{-1}\right)\hat{L}.
\end{aligned}
$$

which proves that these are dynamical symmetries of (1.7). q.e.d.

We now ask whether the finite-dimensional representations (2.19, 2.24, 2.25), with $\mu \neq 0$, acting on functions $f = f(t, r, v)$, and having a dynamical exponent $z \neq 1$, can be extended to representations of an infinite-dimensional conformal Virasoro algebra. The answer turns out to be negative:

Proposition 5: *The representations (2.19, 2.24, 2.25) of the finite-dimensional conformal algebra $\langle X_n, Y_n \rangle_{n \in \{\pm 1, 0\}}$ with commutators (2.1) cannot be extended to representations of an infinite-dimensional conformal Virasoro algebra with commutators (2.1) when $z \neq 1$.*

Similar no-go results have been found before for variants of representations of the Schrödinger and conformal galilean algebras [11]. On the other hand, for $\mu = 0$ extensions to a representation of a conformal Virasoro algebra with $z \neq 1$ exist [3].

Proof: Since for the finite-dimensional representations (2.19, 2.24, 2.25), we have

$$
[X_n, X_{n'}] = (n - n')X_{n+n'}, \quad [X_n, Y_m] = (n - m)Y_{n+n'}, \quad n, n', m = 0, \pm 1
$$

we suppose that this must be valid for all admissible $n, m \in \mathbb{Z}$. Now using the condition (2.10) for $n = 2$, a conformal Virasoro algebra should contain a new generator X_2. Starting from the most general form, $X_2 = -a_2(t, r, v)\partial_t - b_2(t, r, v)\partial_r - c_2(t, r, v)\partial_v - d_2(t, r, v)$ we find that the coefficients are obtained from:

$$
\begin{aligned}
a_2 &= t^3 + a_{21}(r, v), \quad b_2 = \frac{3}{z} t^2 r + 3\frac{z-2}{z} \mu t r^2 v^{-1} + b_{21}(r, v) \\
c_2 &= 3\frac{1-z}{z}(v t^2/2 - \mu r t) + c_{21}(r, v), \quad d_2 = \frac{3}{z} x t^2 - \frac{6}{z}\mu x + d_{21}(r, v),
\end{aligned}
$$

where $a_{21}(r, v), b_{21}(r, v), c_{21}(r, v), d_{21}(r, v)$ are unknown functions of their arguments, but do no longer depend on the time t. We want to satisfy $[X_2, Y_{-1}] = 3Y_1$. However when calculating

$$
\begin{aligned}
[X_2, Y_{-1}] &= [-a_2\partial_t - b_2\partial_r - c_2\partial_v - d_2, -v\partial_r] = \\
&= 3Y_1 - v a_{21}' \partial_t - (3\frac{1-z}{2z} t^2 v - \frac{3}{z}(1-z)\mu t r + v b_{21}' - c_{21} + 3\frac{z-2}{z}\mu^2 r^2 v^{-1})\partial_r \\
&\quad -(v c_{21}' + 3\frac{1-z}{z}\mu(t v - 2\mu r))\partial_v - v d_{21}' - \frac{6}{z}\mu\gamma r v^{-1}
\end{aligned}
$$

we see that closure is not possible for $z \neq 1$. Indeed, although the dependence on r, v of the functions $a_{21}, b_{21}, c_{21}, d_{21}$ can be chosen to satisfy the above closure condition, the t-dependence can not be absorbed into these functions. Hence our new representations (2.19, 2.24, 2.25) of the conformal algebra (2.1) are necessarily finite-dimensional. q.e.d.

3 Symmetry algebra of collisionless Boltzmann equation with an extra force term

We write the collisionless Boltzmann equation in the form

$$\hat{B}f(t,r,v) = (\mu\partial_t + v\partial_r + F(t,r,v)\partial_v)\, f(t,r,v) = 0. \tag{3.1}$$

We want to determine the admissible forms of an external force $F(t,r,v)$ such that the equation (3.1) is invariant under a representation of the conformal algebra (2.1). The unknown representation must include the "force" term and in particular for $F(t,r,v) = 0$ it should coincide with the representations of conformal algebra obtained in previous section.

The idea of the construction is similar to the one used in section 2. First, we impose invariance under basic symmetries:

- From invariance under time-translation $X_{-1} = -\partial_t$, it follows

$$[X_{-1}, \hat{B}] = -\dot{F} = 0 \to F = F(r,v) \tag{3.2}$$

- From invariance under dynamical scaling $X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - \frac{x}{z}$, we obtain that

$$[\hat{B}, X_0] = -\hat{B}, \tag{3.3}$$

  if $F(r,v)$ satisfies the equation $(r\partial_r + (1-z)v\partial_v - (1-2z))F(r,v) = 0$, with solution

$$F(r,v) = r^{1-2z}\varphi(r^{z-1}v), \tag{3.4}$$

  where $\varphi(u)$ is an arbitrary function, of the scaling variable $u := r^{z-1}v$.

It turns out that for the following calculations, it is more convenient to make a change of independent variables $(t,r,v) \mapsto (t,r,u)$. In the new variables, the generator of dynamical scaling just reads:

$$X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{x}{z}. \tag{3.5}$$

Next, in order to be specific, we make the following ansatz for the analogue of space translations[9]

$$Y_{-1} = -r^{1-z}u\partial_r - r^{-z}\Phi(u)\partial_u, \quad \Phi(u) = (z-1)u^2 + \varphi(u). \tag{3.6}$$

In the same coordinate system, the collisionless Boltzmann equation becomes

$$\hat{B}f(t,r,u) = \left(\mu\partial_t + r^{1-z}u\partial_r + r^{-z}\Phi(u)\partial_u\right) f(t,r,u) = 0. \tag{3.7}$$

Here, some comments are in order. In the structure of Boltzmann equation (3.7), as well as in the form (3.6) of the modified space translations Y_{-1}, enters an unknown function $\Phi(t,r,u)$.

[9] Indeed, we might also require to find Y_{-1} from the conditions to be (i) a symmetry of Boltzmann equation and (ii) to form a closed Lie algebra with the other basic symmetries $X_{-1,0}$. Such requirements lead to a system of differential equations and the ansatz (3.6) is a particular solution of this system, which has the special property that the Boltzmann operator can be linearly expressed $\hat{B} = -\mu X_{-1} - Y_{-1}$ by the generators. We believe this to be a natural auxiliary hypothesis.

Therefore, the form of X_1 cannot be found only from its commutator with the other generators X_n, but the constraints form the entire conformal algebra must be used, as well as the requirement that X_1 and $Y_{0,1}$ are dynamical symmetries of eq. (3.7)

$$[\hat{B}, X_1] = \lambda_{X_1}(t,r,v)\hat{B}, \quad [\hat{B}, Y_0] = \lambda_{Y_0}(t,r,v)\hat{B}, \quad [\hat{B}, Y_1] = \lambda_{Y_1}(t,r,v)\hat{B}. \tag{3.8}$$

In fact, commuting the unknown generators X_1, Y_0, Y_1 with X_{-1} and X_0, we can fix the t- and r-dependence of the yet undetermined functions which occur in them

$$\begin{aligned}
Y_0 &= -r^z a_0(u)\partial_t - (r^{1-z}u + rb_0(u))\partial_r - (r^{-z}\Phi(u)t + c_0(u))\partial_u - d_0(u) \\
X_1 &= -(t^2 + r^{2z}a_{12}(u))\partial_t - ((2/z)tr + r^{z+1}b_{12}(u))\partial_r \\
&\quad -r^z c_{12}(u)\partial_u - (2/z)xt - r^z d_{12}(u) \\
Y_1 &= -\left(2tr^z a_0(u) + r^{2z}A(u)\right)\partial_t - \left(t^2 r^{1-z}u + 2trb_0(u) + r^{z+1}B(u)\right)\partial_r \\
&\quad -\left(t^2 r^{-z}\Phi(u) + 2tc_0(u) + r^z C(u)\right)\partial_u + (2/z)\mu xt - r^z D(u),
\end{aligned} \tag{3.9}$$

with the four functions

$$\begin{aligned}
A(u) &= 2zb_0a_{12} + c_0a'_{12} - za_0b_{12} - a'_0c_{12}, \quad C(u) = zb_0c_{12} + c_0c'_{12} - c'_0c_{12} - a_{12}\Phi \\
B(u) &= \frac{2}{z}a_0 + zb_0b_{12} + c_0b'_{12} - ua'_{12} - b'_0c_{12}, \quad D(u) = \frac{2}{z}xa_0 + zb_0d_{12} + c_0d'_{12}.
\end{aligned} \tag{3.10}$$

In particular, looking for a representation of the analog of extended Galilei algebra $\langle X_{-1}, X_0, Y_{-1}, Y_0\rangle$, we find that the unknown functions $a_0(u), b_0(u), c_0(u), d_0(u)$ must satisfy the system

$$zua_0(u) + \Phi(u)a'_0(u) - k = 0 \tag{3.11}$$
$$zub_0(u) + \Phi(u)b'_0(u) - c_0(u) - qu = 0 \tag{3.12}$$
$$\Phi'(u)c_0 - \Phi(u)c'_0(u) + (q - zb_0)\Phi = 0 \tag{3.13}$$
$$\Phi(u)d'_0(u) = 0 \tag{3.14}$$

Because of (3.14), one must distinguish two cases:

1. $\Phi(u) = 0$, when $d_0(u)$ can be arbitrary
2. $\Phi(u) \neq 0$, when $d_0(u) = d_0 =$ cste. is a constant.

In the second case, taking equation (3.12, 3.13) together, we obtain an equation for $b_0(u)$. It is

$$\Phi^2(u)b''_0(u) + zu\Phi(u)b'_0(u) + (2z\Phi(u) - zu\Phi'(u))b_0(u) - 2s\Phi(u) = 0, \tag{3.15}$$

and has in general two independent solution: $b_{01}(u), b_{02}(u)$. It follows that, for a given arbitrary value of $\Phi(u) \neq 0$, we have in general *two distinct realisations* of the analogue of Galilei transformation; and consequently also two realizations of the analogue of the Galilei algebra. By construction, these are Lie algebras of symmetries of the collisionless Boltzmann equation (3.7) (with $\lambda_{Y_0} = -k/\mu = -(\mu + q)$):

$$\begin{aligned}
&[Y_0, X_{-1}] = Y_{-1}, \quad [X_0, X_{-1}] = X_{-1}, \\
&[Y_0, Y_{-1}] = \frac{k - \mu^2}{\mu}Y_{-1} + kX_{-1}.
\end{aligned} \tag{3.16}$$

Next, we include the generators of special conformal transformation X_1 and Y_1 to the extended Galilei algebras (3.16) just constructed. We must also satisfy the other commutators of the conformal algebra (2.1). Furthermore, the generators of the representation we are going to construct are dynamical symmetries of the collisionless Boltzmann equation.[10] We find:

$$\lambda_{X_1}(t,r,u) = -2t - (r^z/\mu)(2zua_{12} + \Phi(u)a'_{12}(u)) = -2t - 2r^z a_0(u)/\mu \tag{3.17}$$

for the eigenvalue and

$$\begin{aligned}
&c_{12}(u) = (2/z)\mu - (u/\mu)(2za_{12}(u) + \Phi(u)a'_{12}(u)) + (2zub_{12} + \Phi(u)b'_{12}(u)) && (3.18)\\
&zuc_{12}(u) + \Phi c'_{12}(u) - c_{12}(u)\Phi'(u) + zb_{12}(u)\Phi(u) - 2c_0(u) = 0 && (3.19)\\
&zud_{12}(u) + \Phi(u)d'_{12}(u) + (2/z)\mu x = 0 && (3.20)\\
&\Phi^2(u)b''_{12}(u) + 3zu\Phi(u)b'_{12}(u) + z[2zu^2 + 3\Phi(u) - 2u\Phi'(u)]b_{12}(u) &&\\
&\quad - (u/\mu)\Phi^2(u)a''_{12}(u) - [3zu^2 + 2\Phi(u)](\Phi/\mu)a'_{12}(u) &&\\
&\quad - [zu^2 + 3\Phi(u) - u\Phi'(u)](2zu/\mu)a_{12}(u) + (2\mu/z)(zu - \Phi'(u)) = 0 && (3.21)\\
&2zua_{12}(u) + \Phi(u)a'_{12}(u) - 2a_0(u) = 0 && (3.22)\\
&2zub_{12}(u) + \Phi(u)b'_{12}(u) - c_{12}(u) - 2b_0(u) = 0 && (3.23)\\
&b_0(u) = (u/\mu)a_0(u) - \mu/z && (3.24)\\
&c_0(u) = (\Phi/\mu)a_0(u) && (3.25)\\
&d_0(u) = \text{cste.} = -\mu x/z. && (3.26)\\
&k_0 = \alpha_0 k = 2k, \quad q_0 = \alpha_0 q = 2q &&\\
&2zuA(u) + \Phi(u)A'(u) - 2qa_0(u) = 0 && (3.27)\\
&2zuB(u) + \Phi(u)B'(u) - C(u) - 2(k/z + qb_0(u)) = 0 && (3.28)\\
&zuC(u) + \Phi(u)C'(u) - \Phi'(u)C(u) + z\Phi(u)B(u) - 2qc_0(u) = 0 && (3.29)\\
&zuD(u) + \Phi(u)D'(u) - (2x/z)(k - \mu q) = 0. && (3.30)\\
&K = k, \quad Q = q && (3.31)\\
&(q - 2zb_0)A(u) - c_0A'(u) + za_0(u)B(u) + a'_0(u)C(u) + ka_{12}(u) - 2a_0^2 = 0 && (3.32)\\
&(q - zb_0)B(u) - c_0B'(u) + uA(u) + b'_0C(u) + kb_{12}(u) - 2a_0(u)b_0(u) = 0 && (3.33)\\
&(q - zb_0 + c'_0(u))C(u) - c_0C'(u) + \Phi(u)A(u) + kc_{12}(u) - 2a_0(u)c_0(u) = 0 && (3.34)\\
&(q - zb_0)D(u) - c_0D'(u) + kd_{12}(u) + \frac{2a_0(u)\mu x}{z} = 0 && (3.35)\\
&2z(b_{12}(u)A(u) - a_{12}(u)B(u)) + c_{12}(u)A'(u) - a'_{12}(u)C(u) + 2a_0(u)a_{12}(u) = 0 && (3.36)\\
&(2/z)A(u) - c_{12}(u)B'(u) + b'_{12}C(u) - 2b_0(u)a_{12}(u) = 0 && (3.37)\\
&(zb_{12}(u) - c'_{12}(u))C(u) + c_{12}C'(u) - zc_{12}(u)B(u) + 2c_0(u)a_{12}(u) = 0 && (3.38)\\
&(2x/z)(\mu a_{12}(u) + A(u)) + zd_{12}(u)B(u) + d'_{12}(u)C(u) &&\\
&\quad - zb_{12}D(u) - c_{12}(u)D'(u) = 0 && (3.39)
\end{aligned}$$

The system of equations (3.11, 3.12, 3.13, 3.18, 3.19, 3.20, 3.21, 3.22, 3.23, 3.24, 3.25, 3.26, 3.27, 3.28, 3.29, 3.30, 3.31, 3.32, 3.33, 3.34, 3.35, 3.36, 3.37, 3.38, 3.39) must give a solution for the unknown functions $a_0(u), b_0(u), c_0(u), d_0(u), a_{12}(u), b_{12}(u), c_{12}(u), d_{12}(u)$. Of course, it is possible that several of the above equations are equivalent. Because of this fact, although the

[10] We use the commutators $[Y_1, Y_0] = KX_1 + QY_1$ and $[Y_1, Y_{-1}] = k_0X_0 + q_0Y_0$ in order to establish a relation between the constants k, q and K, Q, k_0, q_0.

above system might look to be over-determined, we have not yet been able to produce explicit solution without making auxiliary assumption. A classification of all solutions of the above system is left as an open problem. We shall now describe some examples of solutions of this large system.

Example 1: Let $\Phi(u) = 0$. This case seems to be quite simple, provided it is compatible with our system. From equation (3.11) we obtain

$$a_0(u) = \frac{k}{z} u^{-1} \tag{3.40}$$

Using this value of $a_0(u)$ from equations (3.20, 3.21, 3.22, 3.24, 3.25, 3.26), we directly obtain

$$\begin{aligned} b_0 &= \text{cste.} = \frac{k}{z\mu} - \frac{\mu}{z}, \quad c_0(u) = 0, \quad d_0 = \text{cste.} = -\frac{\mu}{z}x, & (3.41)\\ a_{12}(u) &= \frac{k}{z^2}u^{-2}, \quad b_{12}(u) = \frac{1}{\mu z^2}(k-\mu^2)u^{-1}, \quad c_{12}(u) = 0, \quad d_{12}(u) = -\frac{2\mu x}{z^2}u^{-1}. & (3.42) \end{aligned}$$

When we substitute the above results in relations (3.10), we also find

$$\begin{aligned} A(u) &= \frac{k}{\mu z^2}(k-\mu^2)u^{-2}, \quad B(u) = \frac{1}{\mu^2 z^2}\left(k(k-\mu^2)+\mu^4\right)u^{-1} \\ C(u) &= 0, \quad D(u) = \frac{2\mu^2 x}{z^2}u^{-1}. \end{aligned} \tag{3.43}$$

One can now verify that the above results for the functions $a_0(u), b_0(u), c_0(u), d_0(u)$ and $a_{12}(u)$, $b_{12}(u)$,$c_{12}(u), d_{12}(u), A(u), B(u), C(u), D(u)$ satisfy all equations of the above system. Now, we can finally write the algebra generators

$$\begin{aligned} X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{x}{z} \\ X_1 &= -\left(t^2 + \frac{k}{z^2}r^{2z}u^{-2}\right)\partial_t - \left(\frac{2}{z}tr + \frac{k-\mu^2}{z^2\mu}r^{z+1}u^{-1}\right)\partial_r - \frac{2}{z}xt + \frac{2\mu x}{z^2}r^z u^{-1}, \\ Y_{-1} &= -r^{1-z}u\partial_r, \\ Y_0 &= -\frac{k}{z}r^z u^{-1}\partial_t - \left(tr^{1-z}u + \frac{k-\mu^2}{z\mu}r\right)\partial_r + \frac{\mu x}{z}, \\ Y_1 &= -\left(\frac{2k}{z}tr^z u^{-1} + \frac{k(k-\mu^2)}{z^2\mu}r^{2z}u^{-2}\right)\partial_t \\ &\quad -\left(t^2 r^{1-z}u + 2\frac{k-\mu^2}{z\mu}tr + \frac{k(k-\mu^2)+\mu^4}{z^2\mu^2}r^{z+1}u^{-1}\right)\partial_r + \frac{2}{z}\mu xt - \frac{2\mu^2 x}{z^2}r^z u^{-1}. \end{aligned} \tag{3.44}$$

Returning to the original variables via the change $(t, r, u) \mapsto (t, r, v)$, done through the substitutions $u \to r^{z-1}v$ and $\partial_r \to \partial_r + (1-z)r^{-1}v\partial_v$. Finally we have the following representation of a conformal symmetry algebra of the collisionless Boltzmann equation (3.1):

$$
\begin{aligned}
X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - \frac{x}{z} \\
X_1 &= -\left(t^2 + \frac{k}{z^2}r^2v^{-2}\right)\partial_t - \left(\frac{2}{z}tr + \frac{k-\mu^2}{z^2\mu}r^2v^{-1}\right)\partial_r \\
&- (1-z)\left(\frac{2}{z}tv + \frac{k-\mu^2}{z^2\mu}r\right)\partial_v - \frac{2}{z}xt + \frac{2\mu x}{z^2}rv^{-1}, \\
Y_{-1} &= -v\partial_r - (1-z)r^{-1}v^2\partial_v, \\
Y_0 &= -\frac{k}{z}rv^{-1}\partial_t - \left(tv + \frac{k-\mu^2}{z\mu}r\right)\partial_r - (1-z)\left(tr^{-1}v^2 + \frac{k-\mu^2}{z\mu}v\right)\partial_v + \frac{\mu x}{z}, \\
Y_1 &= -\left(\frac{2k}{z}trv^{-1} + \frac{k(k-\mu^2)}{z^2\mu}r^2v^{-2}\right)\partial_t - \left(t^2v + 2\frac{k-\mu^2}{z\mu}tr + \frac{k(k-\mu^2)+\mu^4}{z^2\mu^2}r^2v^{-1}\right)\partial_r \\
& -(1-z)\left(t^2r^{-1}v^2 + 2\frac{k-\mu^2}{z\mu}tv + \frac{k(k-\mu^2)+\mu^4}{z^2\mu^2}r\right)\partial_v + \frac{2}{z}\mu xt - \frac{2\mu^2 x}{z^2}rv^{-1}. \quad (3.45)
\end{aligned}
$$

Proposition 6: *The generators (3.45) close into the following Lie algebra*

$$
\begin{aligned}
&[X_n, X_{n'}] = (n-n')X_{n+n'}, \quad [X_n, Y_m] = (n-m)Y_{n+m} \\
&[Y_m, Y_{m'}] = (m-m')\left(kX_{m+m'} + \frac{k-\mu^2}{\mu}Y_{m+m'}\right), \quad (3.46)
\end{aligned}
$$

for $n, n', m, m' \in \{-1, 0, 1\}$ *and for an arbitrary dynamical exponent* z. *They give a representation of the finite-dimensional conformal algebra, which acts as dynamical symmetry algebra of the Boltzmann equation in the form:*

$$
\hat{B}f(t,r,v) = (\mu\partial_t + v\partial_r + (1-z)r^{-1}v^2\partial_v)f(t,r,v) = 0. \quad (3.47)
$$

Proof: The commutation relations (3.46) are directly checked. From the commutators $[\hat{B}, X_{-1}] = [\hat{B}, Y_{-1}] = 0$ and

$$
\begin{aligned}
&[\hat{B}, X_0] = -\hat{B}, \quad [\hat{B}, X_1] = -2\left(t + \frac{k}{z\mu}rv^{-1}\right)\hat{B} \\
&[\hat{B}, Y_0] = -(k/\mu)\hat{B}, \quad [\hat{B}, Y_1] = -2\left(\frac{k}{\mu}t + \frac{k}{z\mu^2}rv^{-1}\right)\hat{B}.
\end{aligned}
$$

it is seen that they generate dynamical symmetries. q.e.d.

Example 2: Let $k = 0$. In this case, $\Phi(u)$ left arbitrary, which leads to $a_0 = 0$ from (3.11) and

$$
b_0 = \text{cste.} = -\mu/z, \quad c_0 = 0, \quad d_0 = -\mu x/z \quad (3.48)
$$

Then from eq. (3.10), we obtain

$$
\begin{aligned}
A(u) &= -2\mu a_{12}(u), \quad B(u) = -\mu b_{12}(u) - u a_{12}'(u) \\
C(u) &= -\mu c_{12}(u) - a_{12}(u)\Phi(u), \quad D(u) = -\mu d_{12}. \quad (3.49)
\end{aligned}
$$

However, when substituting in eq.(3.32, 3.33, 3.34, 3.35), taking also into account that $q = -\mu$, we find that $A(u) = a_{12}(u) = 0$. Then it is easy to check that the conditions (3.36, 3.37, 3.38,

3.39) are fulfilled. This allows us to formulate

Proposition 7: *Let $\Phi(u) = (z-1)u^2 + \varphi(u)$. Consider the generators*

$$\begin{aligned}
X_{-1} &= -\partial_t, \quad X_0 = -t\partial_t - \frac{r}{z}\partial_r - \frac{1-z}{z}v\partial_v - \frac{x}{z} \\
X_1 &= -t^2\partial_t - \left(\frac{2}{z}tr + r^{z+1}b_{12}(u)\right)\partial_r \\
&- (1-z)\left(\frac{2}{z}tv + r^z v b_{12}(u) + \frac{r^{1-2z}}{1-z}c_{12}(u)\right)\partial_v - \frac{2}{z}xt - r^z d_{12}(u), \\
Y_{-1} &= -v\partial_r - (1-z)\left(r^{-1}v^2 + \frac{r^{1-2z}}{1-z}\Phi(u)\right)\partial_v = -v\partial_r - r^{1-2z}\varphi(u)\partial_v, \\
Y_0 &= -\left(tv - \frac{\mu}{z}r\right)\partial_r - (1-z)\left(\frac{r^{1-2z}}{1-z}\varphi(u)t - \frac{\mu}{z}v\right)\partial_v + \frac{\mu x}{z}, \\
Y_1 &= -\left(t^2 v - 2\frac{\mu}{z}tr - \mu r^{z+1}b_{12}(u)\right)\partial_r + \frac{2}{z}\mu xt + \mu r^z d_{12}(u) \\
&\quad -(1-z)\left(t^2\frac{r^{1-2z}}{1-z}\varphi(u) - \frac{2}{z}\mu tv - n\mu r^z v b_{12}(u) - \mu\frac{r^{1-2z}}{1-z}c_{12}(u)\right)\partial_v,
\end{aligned} \tag{3.50}$$

where $c_{12}(u) = 2zub_{12}(u) + ((z-1)u^2 + \varphi(u))b'_{12}(u) + 2\mu/z$ and $\varphi(u), b_{12}(u), d_{12}(u)$ satisfy

$$\begin{aligned}
[(z-1)u^2 + \varphi(u)]^2 b''_{12}(u) + 3zu[(z-1)u^2 + \varphi(u)]b'_{12}(u) & \\
+z[(z+1)u^2 - 2u\varphi'(u) + 3\varphi(u)]b_{12}(u) + [(2-z)u - \varphi'(u)]2\mu/z &= 0 \qquad (3.51) \\
zud_{12}(u) + [(z-1)u^2 + \varphi(u)]d'_{12}(u) + 2\mu x/z &= 0. \qquad (3.52)
\end{aligned}$$

For any triplet $(\varphi(u), b_{12}(u), d_{12}(u))$ which gives a solution of the system (3.51, 3.52), the generators (3.50) close into the following Lie algebra

$$\begin{aligned}
&[X_n, X_{n'}] = (n-n')X_{n+n'}, \quad [X_n, Y_m] = (n-m)Y_{n+m} \\
&[Y_m, Y_{m'}] = -\mu\,(m-m')Y_{m+m'},
\end{aligned} \tag{3.53}$$

for $n, n', m, m' \in \{-1, 0, 1\}$ and for an arbitrary constant z. Eq. (3.50) is a representation of the finite-dimensional conformal algebra and acts as dynamical symmetry algebra of the Vlasov-Boltzmann equation, with a quite general "force" term:

$$\hat{B}f(t,r,v) = (\mu\partial_t + v\partial_r + r^{1-2z}\varphi(u)\partial_v)f(t,r,v) = 0. \tag{3.54}$$

Proof: The commutators are satisfied for $k = 0$ and $q = -\mu$ if conditions (3.51,3.52) are fulfilled. Under the same conditions, the symmetries are proven by the relations

$$\begin{aligned}
&[\hat{B}, X_{-1}] = [\hat{B}, Y_{-1}] = [\hat{B}, Y_0] = [\hat{B}, Y_1] = 0 \\
&[\hat{B}, X_0] = -\hat{B}, \quad [\hat{B}, X_1] = -2t\hat{B}.
\end{aligned}$$

q.e.d.

In particular, if we implement the physical requirement that the "force" term should depend only on the positions r, that is $\varphi(u) = \varphi_0 = \text{cste.}$, we can compute explicitly the representation

of the algebra (3.50). To do this, one must find a solution of the system

$$\begin{aligned}
[(z-1)u^2+\varphi_0]^2 b_{12}''(u) + 3zu[(z-1)u^2+\varphi_0]b_{12}'(u) & & \\
+z[(z+1)u^2+3\varphi_0]b_{12}(u) + 2\mu\frac{2-z}{z}u &= 0 & (3.55)\\
zud_{12}(u) + [(z-1)u^2+\varphi_0]d_{12}'(u) + 2\mu x/z &= 0. & (3.56)
\end{aligned}$$

The solution of the second equation is relatively simple, even for an arbitrary z

$$d_{12}(u) = -\delta_0[(z-1)u^2+\varphi_0]^{\frac{z}{2(1-z)}} \int_{\mathbb{R}} \mathrm{d}u\, [(z-1)u^2+\varphi_0]^{\frac{z-2}{2(1-z)}}, \quad \delta_0 = \text{cste.} \tag{3.57}$$

The solution of the equation (3.55) for an arbitrary z can be expressed in terms of hypergeometric functions, but we shall not give its explicit form here. However, for $z = 2$, the system (3.55, 3.56) has an elementary solution

$$\begin{aligned}
b_{12}(u) &= b_{120}\frac{u}{(u^2+\varphi_0)^2} + b_{121}\frac{u^2-\varphi_0}{(u^2+\varphi_0)^2}, \quad b_{120} = \text{cste.}, b_{121} = \text{cste.}\\
d_{12}(u) &= -\mu x \frac{u}{u^2+\varphi_0}. \qquad (3.58)
\end{aligned}$$

Substituting this into the generators (3.50) for $z = 2$, gives a finite-dimensional representation of the dynamical conformal symmetry of a collisionless Boltzmann equation of the form

$$\hat{B}f(t,r,v) = (\mu\partial_t + v\partial_r + \varphi_0 r^{-3}\partial_v)f(t,r,v) = 0. \tag{3.59}$$

4 Conclusions

In this work, we have described the results of a first exploration of dynamical symmetries of collisionless Vlasov-Boltzmann transport equations. Our main finding is that these equations admit conformal dynamical symmetries, although it does not seem to be possible to extend this to infinite-dimensional conformal Virasoro symmetries (not even in the case of $d = 1$ space dimensions). These conformal symmetries are new representations of the conformal algebra and are inequivalent to the standard representation which is habitually used in conformal field-theory descriptions of equilibrium critical phenomena. Our first class of new symmetries was found by admitting the momentum p (or equivalently the velocity $v = p/\mu$) as an additional independent variable, leading to the representations (2.19, 2.24, 2.25). The second class of symmetries also allowed for external driving forces $F(t,r,v)$ and it has been one of the questions which types of forces should be compatible with conformal invariance. As an example, we have seen that time-independent forces $F(r,v) = r^{1-2z}\varphi(r^{z-1}v)$, with an arbitrary scaling function φ, are admissible, and lead to the general representation (3.50). However, the solutions of the associated system of equations for the coefficients have not yet been classified and the complete content of these representations remains to be worked out in the future.

Some intuition can be gleaned from some examples. We have written down the explicit representations for the force $F(r,v) = (1-z)r^{-1}v^2$, with an $z > 1$ arbitrary (3.45), and for $F(r,v) = \varphi_0\, r^{1-2z}$ (3.50, 3.55, 3.56) with an arbitrary $z > 1$. In the later case, which could be related to physical situations, we have given the explicit representation of conformal algebra

for $z = 2$, when $F(r, v) = \varphi_0 r^{-3}$ (3.50, 3.58). Having identified these symmetries, the next step would be to use these to find either exact solutions [6] or else to use the algebra representations for fixing the form of co-variant n-point correlation functions, in analogy with time-dependent critical phenomena, see e.g. [12].

The results derived here can be used as a starting point to derive forms of the transition rates w in the collision terms which would be compatible with the dynamical symmetries of the collision-free equations. This kind of approach would be analogous to the one used for finding dynamical symmetries of non-linear Schrödinger equations, see e.g. [2, 16]. We hope to return to this elsewhere.

Acknowledgments Most of the work on this paper was done during the visits of S.S. at Université de Lorraine Nancy and of M.H. at XI$^{\text{th}}$ International workshop *"Lie theories and its applications in physics"*. These visits was supported by PHC Rila. M.H. was partly supported by the Collège Doctoral Nancy-Leipzig-Coventry (Systèmes complexes à l'équilibre et hors équilibre) of UFA-DFH.

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